SECOND ORDER HOMOGENEOUS HAMILTONIAN OPERATORS AND PROJECTIVE GEOMETRY

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In 1983, B. A. Dubrovin and S. P. Novikov studied for the first time Hamiltonian structures for PDEs whose operator is homogeneous in the order of derivation. In particular, they introduced first order homogeneous Hamiltonian operators and, later, higher order operators of the same type. As a natural question, we wonder when a given system of PDEs possesses such a Hamiltonian structure. Necessary conditions for homogeneous quasilinear systems (also known as hydrodynamic type systems) to admit a Hamiltonian structure with homogeneous operators were found by different authors in the last years: for first order operators by S. Tsarev [1] and for third order operators by E. Ferapontov, M. Pavlov and R. Vitolo [2].

In this talk, we present similar conditions for second order homogeneous Hamiltonian operators as discussed in [3, 4]. These operators have the general form

$$P^{ij} = g^{ij}\partial_x^2 + b^{ij}_k u^k_x \partial_x + c^{ij}_k u^k_{xx} + c^{ij}_{kh} u^k_x u^h_x, \tag{1}$$

where $g^{ij}, b^{ij}_k, c^{ij}_k$ and c^{ij}_{kh} are functions depending on the field variables. Surprisly, the conditions found in this case can be solved, giving an explicit form for the conservative systems which admit second order structures.

The resulting systems and the operators turn out to be projective invariant and to possess interesting geometric properties:

Theorem 1. *Second-order homogeneous Hamiltonian operators are invariant under projective reciprocal transformations and the operators of dimension n can be put in bijection with 3-forms in the $n+1$-dimensional space $\mathbb{C}^{n+1}$.*

Finally, the previous result enables us to classify second-order homogeneous Hamiltonian operators in low dimensions ($n \leq 8$).

This is a joint work with Raffaele Vitolo.

References

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