A DISTRIBUTIONAL APPROACH TO FRACTIONAL SOBOLEV SPACES AND FRACTIONAL VARIATION

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We develop a new distributional approach to fractional Sobolev spaces and to a newly-defined *fractional variation*. Given a parameter $\alpha \in (0,1)$, we consider the *fractional Riesz gradient* $\nabla^\alpha = \nabla I_{1-\alpha}$, where I_s is the Riesz potential operator of order s, and study the spaces $S^{\alpha,p}(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : \nabla^\alpha f \in L^p(\mathbb{R}^n;\mathbb{R}^n)\}$, for $p \in [1,+\infty]$, and $BV^\alpha(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : |D^\alpha f|(\mathbb{R}^n) < +\infty\}$, both defined in the usual distributional sense via the *fractional integration-by-parts formula* $\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} \nabla^\alpha f \cdot \varphi \, dx$ valid for all functions $f \in C_c^\infty(\mathbb{R}^n)$ and vector fields $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$. Our distributional approach allows to develop a quite rich and flexible theory, paralleling the known Sobolev–De Giorgi theory in this new fractional framework. We introduce new notions of *fractional Caccioppoli α-perimeter* and of *fractional reduced boundary* and prove several results, such as: a fractional version of De Giorgi's Blow-up Theorem [5]; the identification $S^{\alpha,p} = L^{\alpha,p}$ for $p \in (1,+\infty)$, where $L^{\alpha,p}$ is the usual *fractional Bessel potential space*, and an asymptotic study of the involved fractional operators, both in the pointwise and in the Γ-limit sense, via some new fractional interpolation inequalities [3, 6]; fine properties of BV^α-functions and of the fractional variation [4]; new fractional Leibniz and Gauss–Green formulas [7]. Very recently, our theory has revealed to be a promising field for the study of several new challenging applications of the theory of PDEs and of the Calculus of Variations to fractional operators [1, 2, 8, 9, 10, 11, 12, 13]. This is a research project in collaboration with Giovanni E. Comi. We acknowledge the collaboration of Elia Bruè, Mattia Calzi and Daniel Spector.

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