THE TOPOLOGY OF REAL ALGEBRAIC SETS WITH ISOLATED SINGULARITIES IS DETERMINED BY THE FIELD OF RATIONAL NUMBERS

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The aim of this talk is to describe the topology of real algebraic sets by means of polynomial equations whose coefficients are as simple as possible. In [4] the authors provide an effective procedure to modify the coefficients of a given system of real polynomial equations getting a new system of polynomial equations whose coefficients are real algebraic numbers, while preserving the topology of the starting common solution set. However, when trying to get equations with rational coefficients their result only applies in few cases depending on the starting polynomial equations. Therefore, to investigate the open question about the possibility of describing over $\mathbb{Q}$ the topology of real algebraic sets, we introduce the notion of $\mathbb{Q}$-determined real algebraic set. In particular, $\mathbb{Q}$-determined nonsingular real algebraic sets are in some sense the minimal class of real algebraic sets, in terms of assumptions to be required, to develop new smooth approximation techniques over $\mathbb{Q}$ with regard to [3], [5] and [1]. Then, applying the mentioned new smooth approximation techniques over $\mathbb{Q}$, we get a relative version of the classical Nash-Tognoli's theorem over $\mathbb{Q}$ (see [2]), that is:

Theorem 1. *Every compact $\mathscr{C}^\infty$ manifold $M \subset \mathbb{R}^n$ containing $\mathscr{C}^\infty$ submanifolds M_i of codimension one in general position, for $i = 1, \dots, \ell$, can be arbitrarily $\mathscr{C}^\infty$ approximated by a $\mathbb{Q}$-determined projectively $\mathbb{Q}$-closed nonsingular algebraic set $M' \subset \mathbb{R}^m$, for some $m \geq n$, containing $\mathbb{Q}$-determined nonsingular algebraic subsets M'_i of codimension one in general position, for $i = 1, \dots, \ell$, such that each M'_i approximates M_i, for every $i = 1, \dots, \ell$.*

Moreover, after interpolation techniques, resolution of singularities, applications of the above Theorem 1 and blowing down operations over $\mathbb{Q}$ we are able to get results also in non-compact and singular cases. Indeed, our main result, which i will explain in deep, is the following:

Theorem 2. *Every real algebraic set $V \subset \mathbb{R}^n$ with isolated singularities is semi-algebraically homeomorphic to a $\mathbb{Q}$-determined real algebraic set $V' \subset \mathbb{R}^m$, with $m \geq n$. Furthermore, the homeomorphism $\phi : V \to V'$ we construct has the following additional properties:*

(i) *it preserves nonsingular points and restricts to a Nash diffeomorphism between the nonsingular loci,*
(ii) *it extends to a semi-algebraic homeomorphism from $\mathbb{R}^m$ to $\mathbb{R}^m$,*
(iii) *it is arbitrarily $\mathscr{C}^0$-close to the inclusion map $\iota : \mathbb{R}^n \hookrightarrow \mathbb{R}^m$ on compact subsets of V and arbitrarily $\mathscr{C}^\infty$-close to ι on compact subsets of* $\mathrm{Nonsing}(V)$.

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