EXTREMALS AND CRITICAL POINTS OF THE SOBOLEV INEQUALITY

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The starting point of the talk is the sharp version of the classical Sobolev inequality in $\mathbb{R}^n$ proved in two independent papers: [8] and [1]. The Sobolev inequality has been object of several investigations and generalizations. In particular, in [5] the authors prove a Sobolev-type inequality in $\mathbb{R}^n$ for an anisotropic norm (i.e. a function $H : \mathbb{R}^n \to \mathbb{R}$ convex, positive 1-homogeneous and positive). The proof in [5] is based on the optimal transport technique and leads to the sharp anisotropic Sobolev inequality. In [4] we realize that the optimal transport technique can be used to prove a sharp anisotropic Sobolev-type inequality in convex cones of $\mathbb{R}^n$ (see also [6] and [2] for previous results).

Moreover, an important and well-studied result related to the Sobolev inequality is the classification of critical points, i.e. entire solutions to the so-called critical p-Laplace equation

$$\Delta_p u + u^{p^*-1} = 0 \quad \text{in } \mathbb{R}^n \,, \tag{1}$$

where Δ_p is the usual p-Laplace operator and p^* is the Sobolev critical exponent, explicitly

$$\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u) \quad \text{and} \quad p^* := \frac{np}{n-p} \,.$$

It has been shown (see e.g. [3, 7, 9]), exploiting the moving planes method, that positive solutions to (1) such that $u \in L^p(\mathbb{R}^n)$ and $\nabla u \in L^{p^*}(\mathbb{R}^n)$ can be completely classified. In the talk we will consider the anisotropic critical p-Laplace equation in convex cones of $\mathbb{R}^n$. Since the moving plane method strongly relies on the symmetries of the equation and of the domain, in [4] a different approach to this problem is introduced. In particular, this approach gives a complete classification of the solutions in an anisotropic setting. More precisely, we characterize solutions to the critical p-Laplace equation induced by a smooth norm inside any convex cone of $\mathbb{R}^n$.

The talk is based on the paper [4] in collaboration with G. Ciraolo and A. Figalli.

References

[1] T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*, J. Differential Geom. 11, no. 4, pp. 573–598, 1976.

[2] X. Cabré X. Ros-Oton, J. Serra, *Sharp isoperimetric inequalities via the ABP method*, J. Eur. Math. Soc. 18, no. 12, pp. 2971–2998, 2016.

[3] L. Caffarelli, B. Gidas, J. Spruck, *Asymptotic symmetry and local behaviour of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. 42, pp. 271–297, 1989.

[4] G. Ciraolo, A. Figalli, A. Roncoroni, *Symmetry results for critical anisotropic p-Laplacian equations in convex cones,* Geom. Funct. Anal. 30, pp. 770–803, 2020.

[5] D. Cordero-Erausquin, B. Nazaret, C. Villani, *A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities*, Adv. Math. 182, no. 2, pp. 307–332, 2004.

[6] P.L. Lions, F. Pacella, M. Tricarico, *Best constants in Sobolev inequalities for functions vanishing on some part of the boundary and related questions*, Indiana Univ. Math. J. 37, no. 2, pp. 301–324, 1988.

[7] B. Sciunzi, *Classification of positive $D^{1,p}(\mathbb{R}^n)$-solutions to the critical p-Laplace equation in $\mathbb{R}^n$*, Advances in Mathematics, 291, pp. 12–23, 2016.

[8] G. Talenti, *Best constant in Sobolev inequality*, Ann. di Mat. Pura ed Appl. 110, pp. 353–372, 1976.

[9] J. Vétois, *A priori estimates and application to the symmetry of solutions for critical p-Laplace equations*, J. Differential Equations, 260, pp. 149–161, 2016.

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