RECENT DEVELOPMENTS OF THE SET-THEORETICAL SOLUTIONS TO THE PENTAGON EQUATION

MARZIA MAZZOTTA

The pentagon equation plays an important role in the modern Mathematical Physics, especially in the area of quantum field theory, and it is widely investigated in various contexts. A brief introduction to this topic is contained, for instance, in [3].
Given a set X, a *set-theoretical solution of the pentagon equation*, or briefly a *PE solution*, is a map $s : X \times X \to X \times X$ satisfying the relation

$$s_{23}\, s_{13}\, s_{12} = s_{12}\, s_{23},$$

where $s_{12} = s \times id_X$, $s_{23} = id_X \times s$, and $s_{13} = (id_X \times \tau)s_{12}(id_X \times \tau)$, with $\tau(x, y) = (y, x)$. First examples of PE solutions can be extrapolated in the pioneering work of Kashaev and Sergeev [5], where it is proved that the unique bijective solution on a group G is given by $s(x, y) = (xy, y)$. Recently, our attention has been posed on the study of of this equation, from a purely algebraic point of view [1].

In this talk, we present the complete description of not bijective PE solutions of the form $s\,(x, y) = (xy,\, \theta_x(y))$ on a group G, where θ_x are maps from G into itself, given in [1, Theorem 15]. Such a description involves normal subgroups of the group G, since the set $K = \{x \in G \mid \theta_1(x) = 1\}$ is a normal subgroup of G, despite, in general, θ_1 is not a homomorphism of G.
Moreover, as developed in [2], we show how PE solutions are useful to find set-theoretical solutions of the Yang-Baxter equation, another basic equation of Mathematical Physics, for which in the '90s Drinfel'd [4] posed the problem of finding all its solutions. In particular, we provide a construction of YBE solutions involving PE solutions and specific classes of semigroups by showing that these maps are different from those known until now.

References

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Università del Salento
Email address: marzia.mazzotta@unisalento.it