EQUIVARIANT BIRATIONAL TRANSFORMATIONS AND GENERALIZATIONS OF MATRIX INVERSION

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This is joint work with my supervisor Prof. Solá Conde.

Birational transformations are strongly linked with torus actions, see [4] and the references therein. Going back to the works of Thaddeus [7, 8] and WłWlodarczyk [9], given a birational map ψ between complex projective varieties, there exists an algebraic variety X admitting a $\mathbb{C}^*$-action such that ψ is the map induced among two GIT-quotients of X by $\mathbb{C}^*$.

On the other hand, as pointed out in [3], if we start with a smooth projective variety X admitting a non-trivial $\mathbb{C}^*$-action, then the induced birational transformation ψ between the GIT-quotients can be read out by the properties of the $\mathbb{C}^*$-invariants curves and the Theorem of Białynicki-Birula [1, 2].

We study the birational transformations associated with equalized $\mathbb{C}^*$-actions on a rational homogeneous variety with Picard number one, extending some results of [5].

Theorem 1. *Let G be a simple algebraic group and $X = G/P$ be a rational homogeneous variety of Picard number one. Consider the equalized $\mathbb{C}^*$-action on X determined by a short grading*

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$$

of the Lie algebra of G. Let w_0 be a representative in the class of the longest element in the Weyl group of $\mathfrak{g}$. Suppose also that the action is balanced, i.e. $t^{-1} = w_0 t w_0^{-1}$ for all $t \in \mathbb{C}^*$.

Following [2], let Y_-, Y_+ be sink and source of the action. Then the induced birational map

$$\psi : \mathbb{P}_{Y_-}\left(\mathcal{N}_{Y_-|X}\right) \dashrightarrow \mathbb{P}_{Y_+}\left(\mathcal{N}_{Y_+|X}\right)$$

between the projectivization of normal bundles is completely determined by the inversion map $j : \mathfrak{g}_- \dashrightarrow \mathfrak{g}_-$ of a Jordan algebra as in [6]. Furthermore, if the Lie algebra of P is conjugate to $\mathfrak{g}_- \oplus \mathfrak{g}_0$, then ψ is obtained as the composition of the projectivization of j with the linear isomorphism $\mathrm{Ad}_{w_0} : \mathbb{P}(\mathfrak{g}_-) \longrightarrow \mathbb{P}(\mathfrak{g}_+)$.

We also have a theorem that describes ψ explicitly when $\mathfrak{g}$ is a simple Lie algebra of classic type, which the space in this abstract is too narrow to contain.

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