# A RECENT PERTURBATIVE METHOD TO THE FREE BOUNDARY REGULARITY IN THE ONE-PHASE STEFAN PROBLEM

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In this talk, I will focus on the free boundary regularity in the one-phase Stefan problem

$$\begin{cases} u_t = \Delta u & \text{in } (\Omega \times (0,T]) \cap \{u > 0\}, \\ u_t = |\nabla u|^2 & \text{on } (\Omega \times (0,T]) \cap \partial\{u > 0\}, \end{cases} \tag{1}$$

with $\Omega \subset \mathbb{R}^n$, $u : \Omega \times [0,T] \to \mathbb{R}$, $u \geq 0$. Specifically, I will present a recent approach to the study of the regularity for flat free boundaries of such problem, developed in [7], a joint work with D. De Silva and O. Savin.

In general, in Stefan type problems, free boundaries may not regularize instantaneously. In particular, there exist examples in which Lipschitz free boundaries preserve corners, see for instance [5]. However, in the two-phase Stefan problem, Athanasopoulos, Caffarelli and Salsa showed in [1] that Lipschitz free boundaries in space-time become smooth under a nondegeneracy condition. Moreover, they established the same conclusion in [2] for sufficiently "flat" free boundaries. Their techniques are based on the original work of Caffarelli in the elliptic case [3, 4].

The main result in [7] is essentially equivalent to the previously mention flatness result in [2]. Nevertheless, the method in [7] takes inspiration from the elliptic counterpart established by D. De Silva in [6]. The approach in [7] relies on perturbation arguments leading to a linearization of the problem. In this talk, I will discuss the main steps of such approach, focusing on the main ideas. In particular, I will prove the following result, see [7].

**Theorem 1.** *Fix a constant $K$ (large) and let $u$ be a solution to the one-phase Stefan problem* (1) *in $B_\lambda \times [-K^{-1}\lambda, 0]$ for some $\lambda \leq 1$. Assume that*

$$|u| \leq K\lambda, \qquad u(x_0, t) \geq K^{-1}\lambda \quad \text{for some} \quad x_0 \in B_{\frac{3}{4}\lambda}.$$

*There exists $\varepsilon_0$ depending only on $K$ and $n$ such that if, for each $t$, $\partial_x\{u(\cdot,t) > 0\}$ is $\varepsilon_0$-flat in $B_\lambda$, then the free boundary $\partial\{u > 0\}$ (and $u$ up to the free boundary) is smooth in $B_{\frac{\lambda}{2}} \times [-(2K)^{-1}\lambda, 0]$.*

## References


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