EXTERIOR ALGEBRA: NEW AND OLD RESULTS, OPEN PROBLEMS AND GENERALIZATIONS

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Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. The adjoint action of $\mathfrak{g}$ on itself induces an action of $\mathfrak{g}$ on $\Lambda\mathfrak{g}$ and $S(\mathfrak{g})$, the exterior algebra and the symmetric algebra over $\mathfrak{g}$ respectively.
Some interesting questions about their irreducible decompositions arise naturally:

Q1: Which are the irreducible representations appearing in $S(\mathfrak{g})$ and in $\Lambda\mathfrak{g}$?

Q2: Is it possible to find some formulae for their (graded) multiplicities?

Concerning the symmetric algebra it is proved that the multiplicities are encoded by some special Kazhdan-Luzstig polynomials, determined by the affinization of the Weyl group of $\mathfrak{g}$.
On the other side, despite of the finite dimension (as vector space) of $\Lambda\mathfrak{g}$, determining the irreducible components in the exterior algebra seems to be quite difficult.
Kostant proved (see [6]) that $\Lambda\mathfrak{g}$ is isomorphic, as $\mathfrak{g}$ module, to the direct sum of $2^{\mathrm{rk}\mathfrak{g}}$ copies of the tensor product representation $V_\rho \otimes V_\rho$, where ρ denotes the Weyl vector of $\mathfrak{g}$, with respect to the choice of a set of positive roots in the root system Φ of $\mathfrak{g}$. Unfortunately, this decomposition seems not to be compatible with the grading and links the problem of finding the representations appearing in $\Lambda\mathfrak{g}$ to a Clebsch-Gordan decomposition problem, that is generally hard to be solved. A description of irreducibles in $\Lambda\mathfrak{g}$ it is currently known only in type A and is conjectural for other classical cases (a complete exposition on this topic is contained in [1]).
In the talk I give an overview of some open conjectures and many elegant results proved in the last Century, focusing on graded multiplicities in $\Lambda\mathfrak{g}$ of certain class of irreducible representations. The most famous results in this direction is a theorem due to Hopf, Koszul and Samelson that describes the subalgebra of invariants in $\Lambda\mathfrak{g}$ as an exterior algebra over a suitable set of homogeneous generators of prescribed degrees. Many results on this topic are contained in [7], where Reeder conjectured that it is possible to compute the graded multiplicities of a special class of representations, called "small", reducing to a "Weyl group representations" problem.
We outline the strategy we used to prove Reeder's Conjecture in the classical cases (see [4] and [5]) and we show how our formulae can be rearranged involving the generalized exponents, obtaining a generalization of some well known expressions for graded multiplicities of the adjoint and little adjoint representations in $\Lambda\mathfrak{g}$. As a consequence, we conjecture a generalization of the results contained [3] and [2] to a larger family of special covariants in the exterior algebra.

References

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