PROBABILISTIC VS DETERMINISTIC GAMBLERS

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Is it possible for a gambler using a probabilistic betting strategy to become arbitrarily rich when all gamblers betting according to a deterministic strategy earn only a bounded capital?

We investigate this question in the context of algorithmic randomness, introducing the new notion of *almost everywhere computable randomness*.

Algorithmic randomness aims at formalizing the intuitive concept of randomness for single outcomes, which are usually modelled as infinite binary sequences. A popular way to do so is via the unpredictability approach. We fix a certain class $\mathcal{C}$ of effective gambling strategies for the following game. The bits of an infinite sequence X are revealed in ascending order. When the strategy $B \in \mathcal{C}$ has already seen n many bits of X, B bets a certain amount α of its capital that the $n+1$-th bit of X is, say, 0: if B is right, then B wins α, otherwise B loses α. We say that the strategy B *succeeds* on X if its capital tends to infinity throughout the above infinite game, and we consider a sequence X as *random* (with respect to the given class $\mathcal{C}$) if no betting strategy in $\mathcal{C}$ succeeds on X. In particular, we talk of *computable randomness* when we consider only total computable betting strategies, and of *partial computable randomness* if we also allow partial computable ones. In both cases, however, these strategies are deterministic.

In our framework, instead, we also consider probabilistically effective betting strategies: intuitively speaking, we consider effective betting strategies which, in addition, are allowed to flip a fair coin before placing their bet (and possibly betting accordingly). More formally, we assume that the infinite sequence Y of coin tosses has been drawn in advance and given as an oracle to a partial computable betting strategy B (thus obtaining a partial Y-computable betting strategy which we denote by B^Y): hence, we say that a sequence X is *almost everywhere computably random* if, for any partial computable betting strategy B, we have that

$$\mu\left(\left\{Y : \ B^Y \text{ is total and succeeds on } X\right\}\right) = 0,$$

where μ denotes the uniform measure on the space of infinite binary sequences.

We show that probabilistic betting strategies are in fact stronger than deterministic ones, by building a partial computable random sequence which is not almost everywhere computably random. It is worth noticing that this is an unusual and unexpected result in computability theory, because of a classical theorem stating that every sequence which can be computed by a probabilistic algorithm with positive probability is in fact deterministically computable ([1]).

References

[1] K. de Leeuw, E. F. Moore, C. E. Shannon, N. Shapiro *Computability by probabilistic machines.* Automata studies 34: 183–212, 1956.

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