SURFACES CLOSE TO THE SEVERI LINES

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We are interested in the study of surfaces of general type with maximal Albanese dimension for which the quantity $K_X^2 - 4\chi(\mathcal{O}_X) - 4(q(X) - 2)$ vanishes or is "small" (where $q(X)$ denotes the dimension of the Albanese variety of X). This is related to the Severi inequality which states that such a surfaces satisfies $K_X^2 \geq 4\chi(\mathcal{O}_X)$ (cf. [7]) and equality occurs if and only if the canonical model of X is a double cover of its Albanese variety ramified over an ample divisor (cf. [1]). It is then natural to ask if there is a stronger inequality in which also the irregularity $q(X)$ is involved and this has been achieved by Lu and Zuo ([6]) who have proved that a surface X as above for which $K_X^2 < \frac{9}{2}\chi(\mathcal{O}_X)$ holds, satisfies $K_X^2 \geq 4\chi(\mathcal{O}_X) + 4(q(X) - 2)$.

We show, refining the result of Lu and Zuo, that such a surface with $q(X) \geq 3$ satisfies $K_X^2 = 4\chi(\mathcal{O}_X) + 4(q(X) - 2)$ if and only if its canonical model is a double cover of a product elliptic surface $C \times E$ ramified over an ample divisor $R \sim_{lin} C_1 + C_2 + \sum_{i=1}^{2} dE_i$ with at most negligible singularities. Moreover, if $K_X^2 > 4\chi(\mathcal{O}_X) + 4(q(X) - 2)$ then $K_X^2 \geq 4\chi(\mathcal{O}_X) + 8(q(X) - 2)$ and equality occurs if and only if the canonical model of X is a double cover of an elliptic surface Y without singular fibres ramified over an ample divisor R with at most negligible singularities for which $K_Y.R = 8(q - 2)$.

Because these results are intimately related to the theory of double covers, it is not difficult to see that they hold over any algebraically closed field of characteristic different from 2. Nonetheless, it is possible to prove the same result in characteristic 2 adding some ad hoc hypothesis (e.g. the Albanese morphism is separable).

Some of the results here involved, such as the Severi inequality, have been extended to higher dimensional variety ([3] and [4]): it is then natural to ask, for a possible future work, if it is possible to extend these results for varieties of general type with maximal Albanese dimension of dimension greater than or equal to 3.

References

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