UNITARIZATION AND INVERSION FORMULAE FOR THE RADON TRANSFORM BETWEEN DUAL PAIRS

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The Radon transform has its origin in the problem of recovering a function defined on $\mathbb{R}^d$ from its integrals over hyperplanes. In 1917 Radon proved the reconstruction formula for two and three-dimensional functions. In $\mathbb{R}^3$ it reads

$$f = -\frac{1}{2}\Delta\mathcal{R}^{\#}\mathcal{R}f, \tag{1}$$

where Δ denotes the Laplacian, $\mathcal{R}$ denotes the Radon transform, and $\mathcal{R}^{\#}$ denotes the dual Radon transform, or back-projection. The Radon transform maps a function on $\mathbb{R}^d$ into the set of integrals over all hyperplanes, while the dual Radon transform maps a function defined on the set of hyperplanes of $\mathbb{R}^d$ into its integrals over the sheaves of hyperplanes through a point. This classical inverse problem is a particular case of the more general issue of recovering an unknown function on a manifold by means of its integrals over a family of submanifolds, already investigated by Gelfand in the 1950s. A natural framework for such general inverse problems was considered by Helgason [2]. He introduced the generalized Radon transform associated to dual pairs $(G/K, G/H)$ of homogeneous spaces of the same locally compact group G, where K and H are closed subgroups of G. The space G/K is meant to describe the ambient in which the functions to be analysed live. The second space G/H is meant to parametrise the set of submanifolds of G/K over which one wants to integrate functions. Every element $\xi \in G/H$ defines a subset $\hat{\xi} \subseteq G/K$. The generalized Radon transform $\mathcal{R}$ takes functions on G/K into functions on G/H and is abstractly defined by

$$\mathcal{R}f(\xi) = \int_{\hat{\xi}} f(x)\mathrm{d}m_\xi(x),$$

where, for all $\xi \in G/H$, m_ξ is a suitable measure on the manifold $\hat{\xi}$ and where f is such that the right hand side is meaningful, possibly in some weak sense. In this context, the most relevant issues are to generalize formula (1) and to prove that the Radon transform, up to a composition with a suitable pseudo-differential operator, can be extended to a unitary operator from $L^2(G/K)$ to $L^2(G/H)$. We prove that if the quasi regular representations π of G on $L^2(G/K)$ and $\hat{\pi}$ of G on $L^2(G/H)$ are irreducible, then the generalized Radon transform $\mathcal{R}$, up to composition with a suitable pseudo-differential operator, can be extended to a unitary operator $\mathcal{Q} : L^2(G/K) \to L^2(G/H)$ intertwining the two representations. If, in addition, the representations are square integrable, we derive an inversion formula for the generalized Radon transform based on the voice transform associated to these representations.

This is a joint work with Giovanni S. Alberti, Filippo De Mari and Ernesto De Vito. See [1].

References

[1] G. S. Alberti, F. Bartolucci, F. De Mari, E. De Vito, *Unitarization and Inversion Formulae for the Radon Transform between Dual Pairs*, SIAM J. Math. Anal., 51(6):4356–4381, 2019.

[2] S. Helgason, *The Radon transform*, Progress in Mathematics, vol. 5, 2nd edn. Birkhäuser Boston, Inc., Boston, MA, 1999.

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